Hw 8.

1. Let $p \in \mathbb{R}, p > 0$.

(i) Show that, $\forall x \in \mathbb{R}$, $\exists \bar{m} \in \mathbb{Z}$ such that
$\bar{x} := x + \bar{m}p \in [0, p]$.

(ii) Let $x, \bar{x}, \bar{m}$ be as in (i), and let $|u - x| < p$.
Show that $\bar{u} := u + \bar{m}p$ lies in (at least) one of the three intervals

$$[-p, 0], \ [0, p], \ [p, 2p] \qquad (*)$$

(iii) Let $x, u, \bar{x}, \bar{u}$ be as in (ii), with $|u-x|<p$, and suppose that
$x < u$ and $\bar{u} \notin [0, p]$. Show that

$$\bar{x} < p < \bar{u}.$$

2. Let $f: \mathbb{R} \to \mathbb{R}$ be cts and of period $p > 0$: $f(x+p) = f(x) \ \forall x$.
Show that f is unif. cts in two methods.

(a$_1$) Let $\varepsilon > 0$. By ..., $\exists \delta > 0$ such that

$$|f(x) - f(u)| < \frac{\varepsilon}{2} \quad \forall x, u \in [0, p] \text{ with } |x - u| < \delta. \qquad (1)$$

Hence (?), $\forall m \in \mathbb{Z}$,

$$|f(x) - f(u)| < \frac{\varepsilon}{2} \quad \forall x, u \in [mp, (m+1)p] \text{ with } |x - u| < \delta \qquad (2)$$

(a$_2$) Let $\varepsilon > 0$ and $\delta > 0$ be as in (a$_1$). Define $\bar{\delta} = \min\{\delta, p\}$.
Let $x, u \in \mathbb{R}$ with $x < u$ and $|u - x| < \bar{\delta}$. Then (?)

$$|f(x) - f(u)| = |f(\bar{x}) - f(\bar{u})| < \varepsilon/2 \text{ if } \bar{u} \le p$$

and

$$|f(x) - f(u)| = |f(\bar{x}) - f(\bar{u})| \le |f(\bar{x}) - f(p)| + |f(p) - f(\bar{u})|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ if } \bar{u} \not\le p$$

$\therefore |f(x) - f(u)| < \varepsilon \ \forall x, u \in \mathbb{R}$ with $|x - u| < \bar{\delta}$ (?)

2nd method for Q 2.

(b). Let $\varepsilon > 0$. By ... (applied to $[0, 2p]$ or $[0, p+3]$?)), $\exists\ \delta > 0$ s.t.

$$\left| f(x) - f(u) \right| < \varepsilon \quad \forall\, x, u \in [0, 2p] \xrightarrow{\text{or} \cdots} \text{ with } |x-u| < \delta.$$

Let $\delta^* = \min\{\delta, p\}$ (or $\min\{\delta, 3\}$?)

Let $x, u \in \mathbb{R}$ with $|x-u| < \delta^*$. Show

$$(\#) \quad \left| f(x) - f(u) \right| < \varepsilon.$$

Let $\bar{x}, \bar{u}$ as in Q1, and assume without loss (?) of generality that $x < u$ (and so $\bar{x} < \bar{u}$).

Only two cases to consider: $\bar{u} \in [0, p]$ or $\bar{u} \notin [0, p]$ (so $\bar{u} \in [p, 2p)$ or ... if you used "3")

Then, by ?? and since $|\bar{x} - \bar{u}| < \delta^* \leq \delta$, one has

$$\left| f(\bar{x}) - f(\bar{u}) \right| < \varepsilon,$$

and so (??) (#) is true.

3. Do similarly (in two methods):

$f: [0,\infty) \to \mathbb{R}$ cts and $\lim_{x\to+\infty} f(x) = l \in \mathbb{R}$.

Then f is uniformly cts.

4. Let $f: [0,\infty) \to \mathbb{R}$ cts and $\lim_{x\to+\infty} f(x) = l \in \mathbb{R}$.

Suppose $\exists\ x_0 \in (0,\infty)$ s.t. $f(x_0) > l$.

Show that f attains its maximum:

$\exists\ x^* \in [0,\infty)$ s.t. $f(x^*) \geq f(x)\ \forall x$.

State and prove ~~sim~~ corresponding result for minimum.

5. Let $f, g: \mathbb{R} \to \mathbb{R}$ be unif. cts. Which of the following is true/false (why?)

(i) $f \pm g$ is unif. cts

(ii) $f \cdot g$ is unif. cts

(iii) $x \mapsto f(g(x))$ is unif. cts